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ON THE ITERATED SUSPENSION

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§1

IN THIS paper we consider the question of whether the fiber $W_{n,r}$ of the iterated suspension map $i\colon S^n \to \Omega^r S^{n+r}$ is a loop space. In particular, we will construct, in very many cases, a fibration sequence:

$$S^n \xrightarrow{i} \Omega^r S^{n+r} \xrightarrow{v_{n,r}} B_{n,r}$$

so that $W_{n,r} = \Omega B_{n,r}$. In such a case we will say that $B_{n,r}$ exists. In the following theorem all spaces are localized at a fixed prime p. Let $q = 2(p-1)$.

THEOREM 8.

(a) *If* $p=2$ *and* $r \le n+1$, $B_{n,r}$ *exists.*
(b) *If* $p>2$ *and* $r \le nq+1$, $B_{2n-1,r}$ *exists.*

In these cases we have:

(c) $B_{n,r}$ *is the fiber of a map* $\varphi\colon W_{n+1,r-1} \to S^{(n+1)p-1}$.
(d) *If* $p>3$, $B_{n,r}$ *is an associative* H *space and* $v_{n,r}$ *is an* H *map.*
(e) $S^r(\Omega^r S^{n+r}) \simeq S^r(S^n \times B_{n,r})$.

The case $r=2$ of Theorem 8(a) was conjectured by Mahowald [7]. Selick has also studied this case and has shown [10] that $W_{2p-1,2}$ is a double loop space for $p>2$ and [11] that $W_{2n-1,2}$ is an H space for all p. More details about this example are given in Theorem 9.

This case, for $p>2$ is closely related to a conjecture of Cohen, Moore and Neisendorfer [1]. They constructed maps $\pi_n\colon \Omega^2 S^{2n+1} \to S^{2n-1}$ and conjectured that the loops on the fiber of π_{np} is homotopy equivalent to $W_{2n-1,2}$. Our space $B_{2n-1,2}$ is the fiber of a map φ_n: $\Omega^2 S^{2np+1} \to S^{2np-1}$ but we do not know whether $\varphi_n \sim \pi_{np}$. Conjecture 15 of [4] would follow if this were true.

In Section 2 we will give a clutching construction which yields quasifibrations over mapping cones. This will be applied in Section 3 to construct a fundamental diagram of fibrations. In Sections 4 and 5 we will establish properties in decreasing order of generality, culminating in Theorems 8 and 9.

Throughout this paper we will work with homotopy CW-complexes. We will denote by i any obvious inclusion map, by ω any evaluation map, and by $P_T\colon X \to X/T$ the map pinching the subspace T to a point. We write X^k for the k fold smash product of X with itself. By a fibration sequence we will mean a sequence of spaces and maps homotopy equivalent to that defined by a fibration.

§2

In this section we will define a clutching construction and discuss its properties. We will use this to approximate Hurewicz fibrations. Let $X \cup CA$ be the unreduced mapping cone.

PROPOSITION 1. (a) *Let* $E \xrightarrow{\pi} X$ *be a Hurewicz fibration with fiber* $F = \pi^{-1}(*)$. *Let* θ: $F \times A \to E$ *be a trivialization of* $\pi|_A$; i.e. $\pi\theta(f, a) = a$, *and for each* $a \in A$, θ_a: $F \to \pi^{-1}(a)$ *is a homotopy equivalence. Then there is a quasifibration* $E^\theta \xrightarrow{\pi^\theta} X \cup CA$ *with fiber* F *such that* $\pi^\theta|_X = \pi$; *i.e. there is a pull back diagram* (*Fig.* 1):

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\pi"] & E^\theta \arrow[d, "\pi\theta"] \\ X \arrow[r] & X \cup CA; \end{tikzcd}$$

Fig. 1.

furthermore E^θ/E *is homeomorphic to* $F \rtimes SA$.

(b) *Suppose*† $\bar{E} \xrightarrow{\pi} X \cup CA$ *is a Hurewicz fibration with fiber* $F = \bar{\pi}^{-1}$ (*vertex*) *such that* $\bar{\pi}|_X = \pi$. *Then there is a suitable trivialization* θ *and a mapping* Γ: $E^\theta \to \bar{E}$ *over the identity which is a weak homotopy equivalence* (*Fig.* 2).

$$\begin{tikzcd} E^\theta \arrow[r, "\Gamma"] \arrow[d, "\pi\theta"] & \bar{E} \arrow[d, "\bar{\pi}"] \\ X \cup CA \arrow[r, equal] & X \cup CA \end{tikzcd}$$

Fig. 2.

Finally there is a weak homotopy equivalence between $\bar{E}/E$ *and* $F \rtimes SA$.

Proof. We define E^θ as follows:

$$E^\theta = E \coprod F \times CA/(f, a, 0) \sim \theta(a, f);$$

the projection, π^θ: $E^\theta \to X \cup CA$, is given by $\pi^\theta|_E = \pi$, $\pi^\theta(f, a, t) = (a, t) \in CA \subseteq X \cup CA$, putting 0 at the base of the cone. Write $U_1 = X \cup CA - X$ and $U_2 = X \cup CA - \{\text{vertex}\}$. These sets are open and cover $X \cup CA$. We will show that π^θ is a quasifibration over U_1, U_2 and $U_1 \cap U_2$. Applying [2; Satz 2.2] we conclude that π^θ is a quasifibration. Now $(\pi^\theta)^{-1}(U_1)$ is compatibly homeomorphic with $U_1 \times F$ so π^θ is a quasifibration over U_1 and $U_1 \cap U_2$. There are compatible deformation retractions of U_2 onto X and $(\pi^\theta)^{-1}(U_2)$ onto E. We wish to apply [2; 2.10]. Clearly π^θ is a quasifibration over X since it restricts to π there. Finally the retraction induces θ_a: $F \to \pi^{-1}(a)$ over each point $(a, t) \in U_2$. This completes the proof that π^θ is a quasifibering. Clearly E^θ/E is homeomorphic to $F \times CA/F \times A \equiv F \rtimes SA$.

†See [9; Chapter VII, Theorem 1.1]

To prove (b) we use the homotopy lifting property to construct a map Γ in Fig. 3.

$$\begin{tikzcd}
F \times A \times 1 \arrow[r, "\nu"] \arrow[d, hook] & \bar{E} \arrow[d, "\bar{\pi}"] \\
F \times A \times I \arrow[r, "\mu"'] \arrow[ur, dashed, "\Gamma"] & X \cup CA
\end{tikzcd}$$

Fig. 3.

where $\nu(f, a) = f \in F = \bar{\pi}^{-1}\{\text{vertex}\}$ and $\mu(f, a, t) = (a, t) \in CA \subset X \cup CA$. Define $\theta\colon F \times A \to E$ by $\theta(f, a) = \Gamma(f, a, 0)$; $\theta_a\colon F \to \pi^{-1}(a)$ is just the standard homotopy equivalence between various fibers in a Hurewicz fibration. Thus part (a) applies and Γ defines a map $\Gamma\colon E^\theta \to \bar{E}$ over the identity, where $E \subset E^\theta$ is mapped by the inclusion. Γ is clearly a weak homotopy equivalence by the 5 Lemma. (E^θ, E) has the homotopy extension property, as does $(\bar{E}, E)$ since $(X \cup CA, X)$ does. Thus $\bar{E}/E \simeq \bar{E} \cup CE$ and $E^\theta/E \simeq E^\theta \cup CE$. Thus the induced map $E^\theta/E \equiv F \rtimes SA \to \bar{E}/E$ is a weak homotopy equivalence by [G1; 16.17].

Corollary 2. *The map $p_{E^\theta}\colon E^\theta \cup CE \to SE$ is homotopic to the composition:*

$$E^\theta \cup CE \xrightarrow{P_{CE}} E^\theta/E \simeq F \rtimes SA \xrightarrow{\tilde{\theta}} SE$$

where $\tilde{\theta}(f, t, a) = (t, \theta(f, a))$.

Proof. Define $\varphi\colon E^\theta \cup CE = F \times CA \cup CE \to C^+E \cup C^-E$ by $\varphi(f, t, a) = (t, \theta(f, a)) \in C^+E$, and $\varphi(e, t) = (e, t) \in C^-E$. Then $P_{C^+E}\varphi = P_{E^\theta}$ and $P_{C^-E}\varphi = \tilde{\theta} P_{CE}$ so the result follows since $P_{C^+E} \sim 1 \sim P_{C^-E}$.

§3

In this section we will explain our basic constructions. They will depend on certain *construction data* and will be displayed in a *fundamental diagram*. In particular, this diagram will include a fibration:

$$X(k) \to \Omega Y \to B_k(Y)$$

where $X(k)$ is the fiber of the kth Hopf invariant H_k' constructed in the sequel. H_k' is a generalized Toda–Hopf invariant [5, 8]. Given a space A, we will write A_k for the subspace of the James construction A_∞ generated by words of length $\leq k$ [6].

Construction data: We will assume given spaces X and Y together with:

(a) A map $\alpha\colon (SX)_k \to Y$, and
(b) A map $\beta\colon Y \wedge S^{k-1}X^k \to S^k X^{k+1}$ such that the composition of β with $\alpha|_{SX \wedge 1}$ is homotopic to the map commuting S^{k-1} with X.

Given this data we now describe the *fundamental diagram* (Fig. 4). In this, all horizontal and vertical sequences are fibration sequences. Let $W_k(Y)$ be the fiber of α.

$$\begin{tikzcd}
 & \Omega W_k(Y) \arrow[r, "\Omega\phi"] \arrow[d] & \Omega S^k X^{k+1} \arrow[d, "\simeq"] \\
X(k+1) \arrow[r] \arrow[d] & \Omega((SX)_k) \arrow[r, "H'_{k+1}"] \arrow[d] & \Omega S^k X^{k+1} \arrow[d] \\
\Omega Y \arrow[r, "\simeq"] \arrow[d, "\nu_{k+1}"'] & \Omega Y \arrow[r] \arrow[d] & PS^k X^{k+1} \arrow[d] \\
B_{k+1}(Y) \arrow[r] & W_k(Y) \arrow[r, "\phi"] \arrow[d] & S^k X^{k+1} \\
 & (SX)_k \arrow[d, "\alpha"'] & \\
 & Y &
\end{tikzcd}$$

Fig. 4.

We need to describe the maps φ and H'_{k+1}. By Proposition 1, $W_k(Y) \cup CW_{k-1}(Y) \simeq W_k(Y)/W_{k-1}(Y) \simeq \Omega Y \rtimes (SX)^k$. Define φ as the restriction of the map φ': $W_k \cup CW_{k-1} \to S^k X^{k+1}$ given by the composition:

$$\begin{aligned}
W_k \cup CW_{k-1} &\simeq \Omega Y \rtimes (SX)^k \\
&\xrightarrow{P_{(SX)^k}} \Omega Y \wedge (SX)^k \\
&\xrightarrow{\omega} Y \wedge S^{k-1} X^k \\
&\xrightarrow{\beta} S^k X^{k+1}.
\end{aligned}$$

To construct H'_{k+1} note that $\Omega Y \subset W_{k-1} \subset W_k$ so we may build the diagram (Fig. 5):

$$\begin{tikzcd}
\Omega(SX)_k \arrow[r, "\gamma"] \arrow[d] & \Omega(W_k \cup C\Omega Y) \arrow[r, "\Omega i"] \arrow[d] & (\Omega W_k \cup CW_{k-1}) \arrow[d] \\
\Omega Y \arrow[r] \arrow[d] & P(W_k \cup C\Omega Y) \arrow[r] \arrow[d] & P(W_k \cup CW_{k-1}) \arrow[d] \\
W_k \arrow[r] & W_k \cup C\Omega Y \arrow[r, "i"] & W_k \cup CW_{k-1}
\end{tikzcd}$$

Fig. 5.

where γ is canonically defined by the standard null homotopy $\Omega(SX)_k \to \Omega Y \to W_k$. Now define H'_{k+1} as the composition:

$$\Omega(SX)_k \xrightarrow{\gamma} \Omega(W_k \cup C\Omega Y) \xrightarrow{\Omega i} \Omega(W_k \cup CW_{k-1}) \xrightarrow{\Omega\varphi'} \Omega S^k X^{k+1}.$$

$X(k+1)$ and $B_k(Y)$ are defined as the fibers of H'_{k+1} and φ respectively.

Let

$$\Omega(SX)_k \xrightarrow{i} E_k \to (SX)_{k-1}$$

be the fibration induced from the path space fibration over $(SX)_k$.

PROPOSITION 3. *Suppose given the construction data above. Then:*

(a) H'_{k+1} *depends only on* X, *not on* α, β *or* Y.

(b) H'_{k+1} *factors:*

$$\Omega(SX)_k \xrightarrow{i} E_k \xrightarrow{h_{k+1}} \Omega S^k X^{k+1},$$

and $(h_{k+1})^*$ *maps the image of the cohomology suspension isomorphically onto a direct summand of* $H^*(E_k)$.

(c) *The diagram* (*Fig.* 6):

$$\begin{tikzcd}
\Omega SX \arrow[r, "H'_2"] & \Omega SX \wedge X \\
X \times X \arrow[u, "\mu"] \arrow[r, "P_{X \vee X}"'] & X \wedge X \arrow[u, "i"']
\end{tikzcd}$$

Fig. 6.

homotopy commutes where μ *is the multiplication of loops restricted to* $X \times X$.

Proof. To prove (a) we will compare the map H'_{k+1} given by arbitrary construction data (Y, α, β) to the one given when $Y=(SX)_k$, $\alpha=1$, and β is the natural retraction defined by $(SX)_k \subset (SX)_\infty \simeq \Omega S^2 X$ for $k>1$ and $\beta=1$ in case $k=1$. In the later case W_k is contractible. Thus $W_k \cup C\Omega Y \simeq S\Omega Y$. Furthermore, γ: $\Omega(SX)_k \to \Omega(W_k \cup C\Omega Y) \simeq \Omega S\Omega Y = \Omega S\Omega(SX)_k$ is the adjoint to the identity; to see this recall that in general the adjoint to γ is the coextension defined by the null homotopy $\Omega(SX)_k \to \Omega Y \to W_k$. We now compare this specific situation to the general one in Fig. 7. Here the maps ω and ω' are evaluations, and the vertical maps use α.

$$\begin{tikzcd}
(SX)_k \arrow[r, "\gamma"] & \Omega(W_k \cup C\Omega Y) \arrow[r] & \Omega(W_k \cup CW_{k-1}) \simeq \Omega(\Omega Y \ltimes (SX)^k) \arrow[r, "\omega"] & \Omega(Y \wedge S^{k-1}X^k) \arrow[r, "\omega'"] & \Omega S^k X^{k+1} \\
\Omega(SX)_k \arrow[u, dash, "="] \arrow[r] \arrow[dr] & \Omega S\Omega(SX)_k \arrow[u] \arrow[r] & \Omega SE_k \simeq \Omega(\Omega(SX)_k \ltimes (SX)^k) \arrow[u] \arrow[r, "\omega"] & \Omega((SX)_k \wedge S^{k-1}X^k) \arrow[u, "\Omega(\alpha \wedge 1)"'] \arrow[r, "\Omega\beta'"] & \Omega S^k X^{k+1} \arrow[u, dash, "="'] \\
 & E_k \arrow[ur] & & &
\end{tikzcd}$$

Fig. 7.

The bottom composition is independent of the construction data so part (a) is done. To prove (b), note that the adjoint to the composition $E_k \to \Omega S^k X^{k+1}$ has a right inverse since $SE_k \simeq \Omega(SX)_k \rtimes (SX)^k$. Finally, to prove (c) define $e\colon \Omega SX \times SX \to S\Omega SX$ by $e(w, x, t) = (t, \mu(w, x))$ where $\mu\colon \Omega SX \times X \to \Omega SX$ is the restriction of the multiplication of loops. Since this is the action map which defines the path space fibration, Corollary 2 applies and e is the equivalence occuring in the definition of H_2'. Now define $\lambda\colon X \times X \to \Omega(\Omega SX \times SX)$ by $\lambda(x_1, x_2)(t) = (t, i(x_1), x_2)$. Then $\Omega e \cdot \lambda$ is the adjoint to $\mu\colon X \times X \to \Omega SX$, so $H_2'\mu \sim \Omega\omega \cdot \lambda$. Now $\Omega\omega \cdot \lambda(x_1, x_2)(t) = (x_1, x_2, t)$ and we are done.

It seems likely that the map H_k' constructed here is in fact a desuspension of the map $\bar{H}$ constructed in [5]. Using the fibering $\Omega(X \cup CA) \to E \to X$ and the analysis above, one may construct a map $l\colon \Omega(X \cup CA) \to \Omega((X \cup CA) \wedge A)$ similar to the one used to construct $\bar{H}$.

§4

Throughout this section we will localize at a fixed prime p and set $X = S^{2n-1}$.

PROPOSITION 4.
(a) *For* $k \le p$, $X(k) \simeq S^{2n-1}$.
(b) *For* $k' \le k \le p$, $B_{k'}(Y) \simeq B_k(Y)$ *and the equivalence is compatible with the maps* ν_k, $\nu_{k'}$.
(c) *If* $k' < p$, $W_{k'-1}(Y) \simeq B_{k'} \times S^{2nk'-1}$.

Proof. Consider the Hopf invariant $H_k'\colon \Omega S_{k-1}^{2n} \xrightarrow{i} E_{k-1} \xrightarrow{h} \Omega S^{2nk-1}$. By 3(b), h^* is an isomorphism of $H^{2nk-2}(\Omega S^{2nk-2})$ onto a direct summand of $H^{2nk-2}(E_{k-1})$. From the Serre spectral sequence it is easy to see that all three spaces have isomorphic cohomology in dimension $2nk-2$ and that $H^*(\Omega S_{k-1}^{2n}) \simeq H^*(S^{2n-1}) \otimes H^*(\Omega S^{2nk-1})$. Thus $X(k) \simeq S^{2n-1}$. To prove (b) we consider a diagram (Fig. 8) defined for $k' < k$ where all vertical sequences are

$$\begin{tikzcd}
S^{2n-1} \arrow[r] \arrow[d] & \Omega S_{k'-1}^{2n} \arrow[r] \arrow[d, "\Omega\alpha"] & S^{2n-1} \arrow[r] \arrow[d] & \Omega S_{k-1}^{2n} \arrow[d] \\
\Omega Y \arrow[r, equal] \arrow[d, "\nu_{k'}"] & \Omega Y \arrow[r, equal] \arrow[d] & \Omega Y \arrow[r, equal] \arrow[d, "\nu_k"] & \Omega Y \arrow[d] \\
B_{k'}(Y) \arrow[r] & W_{k-1}'(Y) \arrow[r] & B_k(Y) \arrow[r] & W_{k-1}(Y)
\end{tikzcd}$$

Fig. 8.

fibrations. The composite $B_{k'}(Y) \to B_k(Y)$ is consequently a homotopy equivalence proving (b). This equivalence also provides a splitting for the fibration $B_{k'}(Y) \to W_{k'-1}(Y) \to S^{2nk'-1}$, proving (c).

This argument also can be used to show that $B_{kp^s}(Y) \simeq B_{p^s+1}(Y)$ for $1 < k \le p$, and for $k < p$, $W_{kp^s-1}(Y) \simeq B_{p^s+1} \times S^{2nkp^s-1}$.

We will denote the common homotopy type of $B_k(Y)$ for $1 < k \le p$ as $B(Y)$ and ν_k as ν. With this notation we have:

COROLLARY 5. *Given construction data with* $k < p$, *there is a fibration*

$$S^{2n-1} \to \Omega Y \xrightarrow{\nu} B(Y).$$

PROPOSITION 6. *Suppose Y is an H space, α is the restriction of an H map $\alpha_\infty: S^{2n}_\infty \to Y$, and $p > 3$. Then $B(Y)$ is a homotopy associative H space and v is an H map.*

Proof. We will abbreviate $W_k(Y)$ and $B_k(Y)$ as W_k and B_k respectively. Using α_∞ we construct a commutative diagram (Fig. 9). We define the multiplication on B by

$$\begin{tikzcd}
\Omega Y \times \Omega Y \arrow[r] \arrow[d] & W_k \times W_l \arrow[r] \arrow[d] & S^{2n}_k \times S^{2n}_l \arrow[r] \arrow[d] & Y \times Y \arrow[d] \\
\Omega Y \arrow[r] & W_{k+l} \arrow[r] & S^{2n}_{k+l} \arrow[r] & Y
\end{tikzcd}$$

Fig. 9.

$B_2 \times B_2 \to W_1 \times W_1 \to W_2 \to B_4$. Since the map $W_1 \times W_1 \to W_2$ is the induced inclusion on each axis, this multiplication is the induced equivalence $B_2 \to B_4$ on each axis. The following diagram (Fig. 10) shows that v is an H map: Similarly, a somewhat more complicated diagram

$$\begin{tikzcd}
\Omega Y \times \Omega Y \arrow[rrr] \arrow[d] & & & \Omega Y \arrow[dl] \arrow[d] \\
B_2 \times B_2 \arrow[r] & W_1 \times W_1 \arrow[r] & W_2 \arrow[r] & B_4
\end{tikzcd}$$

Fig. 10.

can be constructed to prove homotopy associativity, based on the associativity of the maps $W_k \times W_l \to W_{k+l}$. This diagram terminates with a map $W_4 \to B_5$ and hence is valid for $p > 5$. To prove associativity for $p = 5$ one must produce a factorization of the composite $B_2 \times B_2 \to W_1 \times W_1 \to W_2$ through B_3. This can be accomplished by showing that the composition with φ: $W_2 \to S^{6n-1}$ is inessential.

Remarks.

1. It is easy to see that an H map $Y \to Y'$ induces an H map $BY \to BY'$.

2. The factorization described in the proof of associativity for $p = 5$ in Proposition 6 can also be used in some cases to show that BY is an H space when $p = 3$. A map $BY \times BY \to BY$ is constructed and it must be shown to be a homotopy equivalence on the axes. This can be done, for example, if BY is atomic.

3. The argument of Proposition 6 can also be used to show that $B_{p^s}(Y)$ is a homotopy associative H space for $p > 3$.

PROPOSITION 7. *Suppose that there is a retraction β': $S^r Y \to S^{2n+r}$ (e.g. $r = 2nk$ and $\beta' = \beta$). Then*

$$S^{r+1}(\Omega Y) \simeq S^{r+1}(S^{2n-1} \times B(Y)).$$

Proof. The argument we give applies generally to a fibration $F \to E \xrightarrow{\pi} B$ with a retraction $S^{r+1} E \xrightarrow{r} S^{r+1} F$. One constructs the equivalence:

$$\begin{aligned}
S^{r+1}(E) \to S^{r+1}(E \times E) &\simeq S^{r+1} E \vee S^{r+1} E \vee S^{r+1}(E \wedge E) \\
&\xrightarrow{u} S^{r+1} F \vee S^{r+1} B \vee S^{r+1}(F \wedge B) \\
&\simeq S^{r+1}(F \times B)
\end{aligned}$$

where $u = r \vee S^{r+1}(\pi) \vee (r \wedge \pi)$.

§5

In this section we will apply the theory developed to the iterated suspension maps for spheres localized at a fixed prime p. Recall the notation from Section 1:

THEOREM 8.

(a) *If $p=2$ and $r\leq n+1$, $B_{n,r}$ exists.*
(b) *If $p>2$ and $r\leq nq+1$, $B_{2n-1,r}$ exists.*

In these cases we have:

(c) *$B_{n,r}$ is the fiber of a map φ: $W_{n+1,r-1}\to S^{(n+1)p-1}$.*
(d) *If $p>3$, $B_{n,r}$ is an associative H space and $\nu_{n,r}$ is an H map.*
(e) *$S^r(\Omega^r S^{n+r})\simeq S^r(S^n\times B_{n,r})$.*

Proof. For (a) we take $X=S^n$, $Y=\Omega^{r-1}S^{n+r}$, α: $S^{n+1}\to\Omega^{r-1}S^{n+r}$, the suspension, and β: $\Omega^{r-1}S^{n+r}\wedge S^n\to S^{2n+1}$ the loop evaluation. By 3(c), $X(2)=S^n$ and let $B_{n,r}=B_2(Y)$. When $r\leq nq$, (b) follows directly from Corollary 5 by taking $X=S^{2n-1}$, $k=p-1$, $Y=\Omega^{r-1}S^{2n+r-1}$, and α and β the obvious maps. The case $r=nq+1$ requires the use of the Serre splitting $\Omega^{nq+1}S^{2np}\simeq\Omega^{nq}S^{2np-1}\times\Omega^{nq+1}S^{4np-1}$. Here we take $B_{2n-1,nq+1}=B_{2n-1,nq}\times\Omega^{nq+1}S^{4np-1}$. (c) follows directly from the fundamental diagram, (d) from Proposition 6 and (e) from Proposition 7.

Remark. One can similarly define fibrations:

$$S^{2n-1}\to\Omega^r S^{2n+r-1}_{(p-1)}\to\bar{B}_{2n-1,r}$$

where $r\leq nq$. By naturality we have a sequence of H spaces and H maps:

$$\Omega S^{2np-1}\to B_{2n-1,2}\to\bar{B}_{2n-1,2}\to B_{2n-1,3}\to\cdots\to B_{2n-1,nq+1}.$$

We now pay particular attention to $B_{2n-1,2}$ which we abbreviate as B_n.

THEOREM 9. *There are fibrations localized at p:*

$$S^{2n-1}\xrightarrow{i}\Omega^2 S^{2n+1}\xrightarrow{\nu_n}B_n$$

$$B_n\xrightarrow{j}\Omega^2 S^{2np+1}\xrightarrow{\varphi}S^{2np-1}$$

where i is the suspension, $j\cdot\nu_n$: $\Omega^2S^{2n+1}\to\Omega^2S^{2np+1}$ is the loops on the pth *James Hopf invariant* [6] *and φ has degree p in dimension $2np-1$. Furthermore:*

(a) *If $p>3$, B_n is an associative H space and the maps ν_n and j are H maps.*
(b) *If $p>2$ there is a map ξ: $\Omega S^{2np-1}\to\Omega^3S^{2np+1}$ of degree 1 such that $\xi\cdot\Omega\varphi\simeq\Omega^3(p\iota)$.*
(c) *$\Omega\varphi=H\cdot P$: $\Omega^3S^{2np+1}\to\Omega S^{2n}_{(p-1)}\to\Omega S^{2np-1}$ for $p\geq 2$.*

Proof. We must show that j is an H map. Consider the diagram (Fig. 11). The left hand square and outside rectangle commute. Thus the difference between the sides on the right

$$\begin{tikzcd}
\Omega^2S^{2n+1}\times\Omega^2S^{2n+1} \arrow[r, "\nu\times\nu"] \arrow[d] & B_n\times B_n \arrow[r, "j\times j"] \arrow[d] & \Omega^2S^{2np+1}\times\Omega^2S^{2np+1} \arrow[d] \\
\Omega^2S^{2n+1} \arrow[r, "\nu"] & B_n \arrow[r, "j"] & \Omega^2S^{2np+1}
\end{tikzcd}$$

Fig. 11.

hand square is annihilated by $v \times v$. However $S^2(v \times v)$ has a right inverse by 8(e). Thus the adjoint of the difference is null homotopic and the diagram commutes. To prove (b) Note that $\Omega H_p: \Omega^2 S^{2n+1} \to \Omega^2 S^{2np+1}$ has order p (see for example [4; Sec. 4]). It follows that the adjoint composition:

$$S^2\Omega^2 S^{2n+1} \xrightarrow{S^2(v_n)} S^2 B_n \xrightarrow{j^*} S^{2np+1}$$

has order p. Since $S^2(v_n)$ has a right inverse, j^* has order p and hence j has order p. We construct ξ as a colifting (Fig. 12): (c) follows from the fundamental diagram.

$$\begin{tikzcd}
\Omega S^{2np-1} \arrow[r, "\xi"] \arrow[d] & \Omega^3 S^{2np+1} \arrow[d] \\
B_n \arrow[r] \arrow[d] & * \arrow[d] \\
\Omega^3 S^{2np+1} \arrow[r, "\Omega(p\iota)"] & \Omega^3 S^{2np+1}
\end{tikzcd}$$

Fig. 12.

Remarks.

1. Cohen, Moore and Neisendorfer [1] construct maps $\pi_n: \Omega^2 S^{2n+1} \to S^{2n-1}$ for $p > 2$ and conjecture that the fiber of π_{np} serves as $B_{n,2}$. This leads one to inquire whether $\varphi_n \sim \pi_{np}$. In particular, we wonder whether $\iota \cdot \varphi_n \sim \Omega^2(p\iota): \Omega^2 S^{2np+1} \to \Omega^2 S^{2np+1}$. This property is enjoyed by π_{np}; conjecture 15 of [4] is equivalent to the statement that $\Omega\iota \cdot \Omega\varphi_n \sim \Omega^3(p\iota)$. We think of 9(b) as a partial attempt at proving this. We also conjecture that φ_n is an H map (when $p > 2$); this is true in case $n = 1$.

2. The above analysis gives a description of the "Whitehead product" map

$$\omega_n: \Omega^2 S^{2np-1} \to S^{2n-1}$$

for $p > 2$. It is the composition:

$$\Omega^2 S^{2np-1} \xrightarrow{\Omega\rho} \Omega S^{2n} \xrightarrow{\pi} S^{2n-1}$$

where π is the H space retraction map and ρ is the composition:

$$\Omega S^{2np-1} \to B_n \xrightarrow{\gamma} W_{2n,1} \xrightarrow{i} S^{2n}$$

where γ is the inclusion of the fiber in the (split) fibration $B_n \to W_{2n,1} \to S^{4n-1}$.

3. The fact that for $p > 3$, $B_{n,r}$ is an H space suggests that some of them may be loop spaces. In fact, $B_{n,1}$ is a loop space and Selick [10] has shown that $W_{2p-1,2}$ is a double loop space for $p > 2$. We are unable to show that his construction for $B_{2p-1,2}$ agrees with ours, or even construct the map v for his space. His methods do suggest searching for a map $\Omega S^{2np+1}\{p\} \xrightarrow{\gamma_n} B_{np}$ which restricts to $\gamma_{np}: \Omega^2 S^{2np+1} \to B_{np}$. If F_{np} is the fiber of such a map, then ΩF_{np} is the fiber of π_{np}.

4. The space $B_{2n,2}$ has the same cohomology as $\Omega^2 S^{4n+3} \times \Omega S^{4n+1}$ and we conjecture that these spaces have the same homotopy type. This means that in this case $\varphi = *$ which has implications for the EHP sequence.

5. It is not at all clear what happens for larger values of r than those covered by 8(b). One can construct fibrations using $k = p^s - 1$ for $s > 1$. In this case the fiber is no longer a sphere. It is the fiber of $\Omega^2 S^{2n+1} \to B_{np^{s}-1}$.

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